

On light-like extremal surfaces in curved spacetimes*

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Abstract

In this paper, we are concerned with light-like extremal surfaces in curved spacetimes. It is interesting to find that under a diffeomorphic transformation of variables, the light-like extremal surfaces can be described by a system of nonlinear geodesic equations. Particularly, we investigate the light-like extremal surfaces in Schwarzschild spacetime in detail and some new special solutions are derived systematically with aim to comparing with the known results and illustrating the method.

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1 Introduction

In recent years the string and membrane theory has drawn great interest. The reason lies in that this theory is not only a possible unification model, but also it tightly relates to extremal surfaces in physical spacetimes from mathematical point of view. For the theory of extremal surfaces in flat Minkowski space-time, there have been many deep results such as Calabi [4] and Cheng and Yau [6] for space-like case, the papers [1], [2], [19], [24] for time-like case, and [11]-[12] for the case of mixed type. Hoppe *et al.* [3] derive the equation for a classical relativistic membrane moving in Minkowski space \mathbb{R}^{1+3} and give some classical solutions, see also [16]. Lindblad [23] studies this equation and proves the global existence of smooth solution for small initial data.

Recently, in the papers [17]-[21] Kong *et al.* restudy the dynamics of relativistic string in Minkowski space \mathbb{R}^{1+n} . Based on the geometric properties enjoyed by the extremal surface equations, they give a sufficient and necessary condition for global existence of extremal surfaces without space-like point in \mathbb{R}^{1+n} with given initial data. Moreover, numerical analysis shows that various topological singularities will develop in finite time in the motion of a string. Surprisingly, they obtain a general solution formula for the highly coupled nonlinear equations. With the aid of this solution formula, they succeed in proving that the motion of a closed string is always time-periodic. For the theory of extremal surfaces in curved spacetimes, there are very few results so far (see e.g., [10]). For a relativistic string, He and Kong [13]-[14] analyze the basic governing equations (see (1) or (2) below) and their inherent interesting properties. Under suitable assumptions, they also provide some positive results on the global existence for the motion of relativistic strings in a general curved spacetime.

For light-like extremal surfaces, it is well-known that the induced metrics are degenerate by definition such that it is not clear how to define the mean curvature and what kind of light-like surfaces can be treated as extremal (cf. [9], [15]). However, we would like to point out that the system (2) below is still valid to describe light-like surfaces. Kong *et al.* [21] give some remarks on light-like extremal surfaces in flat Minkowski space. In [17], Huang and Kong consider three-dimensional light-like extremal sub-manifolds in Minkowski spacetime and present some explicit examples. Gorkavyy [9] distinguishes two particular classes of light-like surfaces in Minkowski spacetime by applying one particular deformability property of surfaces. He [15] studies the light-like extremal surfaces in flat

Minkowski spacetime $\mathbb{R}^{1+(1+n)}$ and gives a necessary and sufficient condition to obtain explicit solution formulas. In physics, another topic related to light-like surfaces is the so-called *null string*, which was introduced by Schild [25] (see also [18]) and later developed by Dabrowski and Larsen [7] in curved spacetimes. In [7], the equations for a null string and the associated constraints in Schwarzschild spacetime are derived and the dynamics for some special solutions are also discussed.

This paper is devoted to investigating light-like extremal surfaces in a general curved spacetime. Here by light-like extremal surfaces, we mean that for such surfaces, they not only satisfy the light-likeness assumption (see (6) below), but also they can be described by the equations (2). It is the light-likeness condition that we are able to simplify the governing equations (2) into a system of geodesic equations in curved spacetimes. Then based on the geodesic equations, we particularly study the light-like extremal surfaces in Schwarzschild spacetime and derive some special solutions to illustrate the method in the present paper.

The reminder of paper is organized as follows. In Section 2, we introduce the equations for light-like extremal surfaces in curved spacetimes. By the light-likeness assumption and a diffeomorphic transformation of variables, we succeed in simplifying these equations and obtaining a system of geodesic equations. Section 3 is devoted to concerning the light-like extremal surfaces in Schwarzschild spacetime. Many examples are provided systematically, which have verified the method presented in this paper for studying the light-like extremal surfaces. Finally, conclusions and some important remarks are given in Section 4.

2 Light-like extremal surfaces in curved spacetimes

In this section, we concern the equations for light-like extremal surfaces in a general curved spacetime (\mathcal{N}, \tilde{g}) , which is a Lorentzian manifold. Some properties enjoyed by these equations are also discussed.

For a two-dimensional extremal surface, denoted by S , the local coordinates are supposed to be (ζ^0, ζ^1) and sometimes for simplicity, we denote $\zeta^0 = t$, $\zeta^1 = \theta$. If the extremal surface is time-like, then the corresponding Euler-Lagrange equations read

$$g^{\alpha\beta} \left(x_{\alpha\beta}^\mu + \tilde{\Gamma}_{\lambda\nu}^\mu x_\alpha^\lambda x_\beta^\nu \right) = 0, \quad (\mu = 0, 1, \dots, n) \quad (1)$$

where $x_\alpha^\mu = \frac{\partial x^\mu}{\partial \zeta^\alpha}$, $x_{\alpha\beta}^\mu = \frac{\partial^2 x^\mu}{\partial \zeta^\alpha \partial \zeta^\beta}$, $\alpha, \beta = 0, 1$, $\lambda, \mu, \nu, \rho = 0, 1, \dots, n$. $g^{\alpha\beta}$ is the inverse of

the induced metric $g_{\alpha\beta} = \tilde{g}_{\mu\nu} x_\alpha^\mu x_\beta^\nu$ on the extremal surface S . The coordinates $x(t, \theta) = (x^0(t, \theta), x^1(t, \theta), \dots, x^n(t, \theta))^T$ describe the surface S and $\tilde{\Gamma}_{\lambda\nu}^\mu$ stand for the connections of the ambient metric \tilde{g} . The above equations can be rewritten in the following form

$$g_{11}x_{tt}^\mu - 2g_{01}x_{t\theta}^\mu + g_{00}x_{\theta\theta}^\mu + g_{11}\tilde{\Gamma}_{\nu\rho}^\mu x_t^\nu x_t^\rho - 2g_{01}\tilde{\Gamma}_{\nu\rho}^\mu x_t^\nu x_\theta^\rho + g_{00}\tilde{\Gamma}_{\nu\rho}^\mu x_\theta^\nu x_\theta^\rho = 0, \quad (2)$$

see for example [14] for details. Based on geometric properties for (2), He and Kong [14] prove a small-data global result for a string moving in general curved spacetimes.

In this paper, we consider the Cauchy problem for (2) with the following initial data

$$t = 0 : \quad x^\mu = \varphi^\mu(\theta), \quad x_t^\mu = \psi^\mu(\theta) \quad (\mu = 0, 1, \dots, n), \quad (3)$$

where $\varphi^\mu(\theta)$ are C^2 -smooth functions with bounded C^2 norm, while $\psi^\mu(\theta)$ are C^1 -smooth functions with bounded C^1 norm. In physics, $\varphi(\theta) = (\varphi^0(\theta), \varphi^1(\theta), \dots, \varphi^n(\theta))$ and $\psi(\theta) = (\psi^0(\theta), \psi^1(\theta), \dots, \psi^n(\theta))$ stand for the initial position and initial velocity of a relativistic string, respectively. Moreover, $\varphi(\theta)$ and $\psi(\theta)$ satisfy the following light-likeness condition

$$(g_{01}[\varphi, \psi](\theta))^2 - g_{00}[\varphi, \psi](\theta)g_{11}[\varphi, \psi](\theta) \equiv 0, \quad (4)$$

in which

$$g_{00}[\varphi, \psi](\theta) \stackrel{\text{def}}{=} \tilde{g}_{\mu\nu}(\varphi)\psi^\mu\psi^\nu, \quad g_{01}[\varphi, \psi](\theta) \stackrel{\text{def}}{=} \tilde{g}_{\mu\nu}(\varphi)\psi^\mu\varphi_\theta^\nu, \quad g_{11}[\varphi, \psi](\theta) \stackrel{\text{def}}{=} \tilde{g}_{\mu\nu}(\varphi)\varphi_\theta^\mu\varphi_\theta^\nu.$$

Now we denote

$$\Delta = \Delta(t, \theta) \stackrel{\text{def}}{=} g_{01}^2 - g_{00}g_{11}, \quad (5)$$

and introduce the following definition for light-like extremal surfaces, which should be due to Kong *et al.* [21] essentially.

Definition 2.1. *Given an ambient curved spacetime (\mathcal{N}, \tilde{g}) , a surface S described by $x(t, \theta) = (x^0(t, \theta), x^1(t, \theta), \dots, x^n(t, \theta))^T$ is called light-like extremal, if $x(t, \theta)$ satisfies the equations (2) and the following light-likeness assumption*

$$\Delta \equiv 0. \quad (6)$$

Remark 2.1. *In fact, for a surface S in curved spacetimes, if $\Delta > 0$ at every point in S , then the surface is said to be entire time-like; if $\Delta < 0$ at every point in S , then the surface is entire space-like; if $\Delta \equiv 0$ at every point in S , then the surface is said to be entire light-like; if a connected surface contains both a time-like part and a space-like part simultaneously, then it is of mixed type.*

By the above remark, we can see that the meaning of light-like extremal surfaces here requires the equations (2) should hold additionally on the light-like surfaces. This method to handle light-like surfaces shall bring us much more convenience in the following discussion.

Remark 2.2. *By making use of the system (2) and the formulas for $\tilde{\Gamma}_{\nu\rho}^\mu$, straightforward computations show that*

$$g_{11}\Delta_t - g_{01}\Delta_\theta = 2\Delta \left(\frac{\partial g_{11}}{\partial t} - \frac{\partial g_{01}}{\partial \theta} \right), \quad (7)$$

which implies the compatibility between the system (2) and the light-likeness condition (6).

Introduce

$$\lambda(t, \theta) \stackrel{\text{def}}{=} -\frac{g_{01}}{g_{11}}, \quad (8)$$

then the system (2) can be rewritten in the following form

$$x_{tt}^\mu + 2\lambda x_{t\theta}^\mu + \lambda^2 x_{\theta\theta}^\mu + \tilde{\Gamma}_{\nu\rho}^\mu(x) (x_t^\nu x_t^\rho + 2\lambda x_t^\nu x_\theta^\rho + \lambda^2 x_\theta^\nu x_\theta^\rho) = 0. \quad (9)$$

First, we have

Lemma 2.1. *Under the assumption (6), we claim that $\lambda = \lambda(t, \theta)$ satisfies the following Burgers equation*

$$\lambda_t + \lambda\lambda_\theta = 0 \quad (10)$$

on the existence domain of smooth solution $x(t, \theta)$.

Proof. By the light-likeness condition (6), direct computation gives

$$\begin{aligned} & \lambda_t + \lambda\lambda_\theta \\ = & \lambda_t + \left(\frac{\lambda^2}{2} \right)_\theta = \left(-\frac{g_{01}}{g_{11}} \right)_t + \left(\frac{1}{2} \frac{g_{00}}{g_{11}} \right)_\theta \\ = & -\frac{1}{g_{11}^2} \left(\frac{\partial g_{01}}{\partial t} g_{11} - g_{01} \frac{\partial g_{11}}{\partial t} \right) + \frac{1}{2g_{11}^2} \left(g_{11} \frac{\partial g_{00}}{\partial \theta} - g_{00} \frac{\partial g_{11}}{\partial \theta} \right) \\ = & \frac{1}{g_{11}^2} \left[g_{01} \left(\frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\rho x_\theta^\mu x_\theta^\nu + 2\tilde{g}_{\mu\nu} x_{t\theta}^\mu x_\theta^\nu \right) \right. \\ & - g_{11} \left(\frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\rho x_t^\mu x_\theta^\nu + \tilde{g}_{\mu\nu} x_{tt}^\mu x_\theta^\nu + \tilde{g}_{\mu\nu} x_t^\mu x_{\theta t}^\nu \right) \\ & + \frac{1}{2} g_{11} \left(\frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_\theta^\rho x_t^\mu x_t^\nu + 2\tilde{g}_{\mu\nu} x_{t\theta}^\mu x_t^\nu \right) \\ & \left. - \frac{1}{2} g_{00} \left(\frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_\theta^\rho x_\theta^\mu x_\theta^\nu + 2\tilde{g}_{\mu\nu} x_{\theta\theta}^\mu x_\theta^\nu \right) \right] \\ = & -\frac{1}{g_{11}} \left[\tilde{g}_{\mu\nu} x_\theta^\nu (x_{tt}^\mu + 2\lambda x_{t\theta}^\mu + \lambda^2 x_{\theta\theta}^\mu) + \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\mu \left(x_t^\rho x_\theta^\nu - \frac{1}{2} x_\theta^\rho x_t^\nu \right) \right. \\ & \left. + \lambda \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\rho x_\theta^\mu x_\theta^\nu + \frac{\lambda^2}{2} \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_\theta^\rho x_\theta^\mu x_\theta^\nu \right]. \quad (11) \end{aligned}$$

By utilizing (2) it follows from (11) that

$$\begin{aligned}
& \lambda_t + \lambda \lambda_\theta \\
&= -\frac{1}{g_{11}} \left[-x_\theta^\nu \tilde{g}_{\mu\nu} \left(\tilde{\Gamma}_{\lambda\rho}^\mu x_t^\lambda x_t^\rho + 2\lambda \tilde{\Gamma}_{\lambda\rho}^\mu x_t^\lambda x_\theta^\rho + \lambda^2 \tilde{\Gamma}_{\lambda\rho}^\mu x_\theta^\lambda x_\theta^\rho \right) \right. \\
&\quad \left. + \frac{\tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\mu \left(x_t^\rho x_\theta^\nu - \frac{1}{2} x_\theta^\rho x_t^\nu \right) + \lambda \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\rho x_\theta^\mu x_\theta^\nu + \frac{\lambda^2}{2} \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_\theta^\rho x_\theta^\mu x_\theta^\nu \right] \\
&= -\frac{1}{g_{11}} \left[-\frac{1}{2} x_\theta^\nu \left(\frac{\partial \tilde{g}_{\nu\rho}}{\partial x^\lambda} + \frac{\partial \tilde{g}_{\nu\lambda}}{\partial x^\rho} - \frac{\partial \tilde{g}_{\lambda\rho}}{\partial x^\nu} \right) \left(x_t^\lambda x_t^\rho + 2\lambda x_t^\lambda x_\theta^\rho + \lambda^2 x_\theta^\lambda x_\theta^\rho \right) \right. \\
&\quad \left. + \frac{\tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\mu \left(x_t^\rho x_\theta^\nu - \frac{1}{2} x_\theta^\rho x_t^\nu \right) + \lambda \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_t^\rho x_\theta^\mu x_\theta^\nu + \frac{\lambda^2}{2} \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\rho} x_\theta^\rho x_\theta^\mu x_\theta^\nu \right] \\
&= 0.
\end{aligned}$$

Thus, the proof is completed. \blacksquare

Remark 2.3. We would like to point out that the method in [14] can not be directly applied here due to the light-likeness condition $\Delta = 0$ (see Theorem 2.1 in [14]).

We consider the Burgers equation (10) associated with the following initial data

$$t = 0 : \quad \lambda = \Lambda(\theta) \stackrel{\text{def}}{=} -\frac{g_{01}[\varphi, \psi](\theta)}{g_{11}[\varphi, \psi](\theta)}. \quad (12)$$

It is well-known that in order the global existence of smooth solutions for the Cauchy problem (10) and (12) exists, we can impose the following sufficient and necessary condition on the initial data (cf. [22]):

$$\Lambda'(\theta) \geq 0, \quad \forall \theta \in \mathbb{R}. \quad (13)$$

Under the assumption (13), the smooth solution $\lambda(t, \theta)$ then can be solved as

$$\lambda(t, \theta) = \Lambda(\vartheta(t, \theta)), \quad (14)$$

where $\vartheta(t, \theta)$ is the inverse function of

$$\theta = \vartheta + \Lambda(\vartheta)t \quad (15)$$

for any fixed $t \geq 0$.

Now we are ready to introduce the following transformation of the variables

$$(t, \theta) \rightarrow (t, \vartheta), \quad (16)$$

where $\vartheta = \vartheta(t, \theta)$ is defined through the equation (15) implicitly.

We have

Lemma 2.2. *Under the assumption (13), the mapping defined by (16) is globally diffeomorphic.*

Proof. It is obvious that the mapping (16) defined through (15) is well-defined on $\mathbb{R}^+ \times \mathbb{R}$. Moreover, under the assumption (13), we have

$$\mathcal{J} \stackrel{\text{def}}{=} \frac{\partial(t, \vartheta)}{\partial(t, \theta)} = \begin{vmatrix} 1 & 0 \\ \frac{\partial \vartheta}{\partial t} & \frac{\partial \vartheta}{\partial \theta} \end{vmatrix} = \frac{\partial \vartheta(t, \theta)}{\partial \theta} = \frac{1}{1 + \Lambda'(\vartheta)t} > 0, \quad (17)$$

for every $(t, \theta) \in \mathbb{R}^+ \times \mathbb{R}$. Then by the Hadamard's Lemma [8], we can conclude that the mapping (16) is globally diffeomorphic. Thus, we complete the proof. ■

Furthermore, direct calculation shows that under the new coordinates (t, ϑ) , the system (9) or (2) can be equivalently reduced into the following form,

$$y_{tt}^\mu + \tilde{\Gamma}_{\nu\rho}^\mu(y) y_t^\nu y_t^\rho = 0, \quad (\mu = 0, 1, \dots, n) \quad (18)$$

where $y^\mu = y^\mu(t, \vartheta) = x^\mu(t, \theta(t, \vartheta))$. Due to $\vartheta = \theta$ at $t = 0$, the initial data for the new system (18) take the following form

$$t = 0 : \quad y^\mu = \varphi^\mu(\vartheta), \quad y_t^\mu = \psi^\mu(\vartheta) \quad (\mu = 0, 1, \dots, n), \quad (19)$$

and the assumption (13) is equivalent to

$$\Lambda'(\vartheta) \geq 0. \quad (20)$$

In addition, it is easy to see that the light-likeness condition (6) can be preserved for this kind of diffeomorphic transformation since we have

$$\Delta(t, \theta) = \Delta(t, \vartheta) \vartheta_\theta^2, \quad (21)$$

where $\vartheta_\theta > 0$ under the assumption (13) or (20).

It is interesting to see that in the new coordinates (t, ϑ) , the system (18) is independent of the variable ϑ . This means that for any point ϑ in the initial curve, the trajectory of this point must be geodesic in the background spacetime. In other words, if the initial data (3) satisfy the light-likeness condition (4) and the assumption (13), all t -curves of the light-like extremal surfaces should be geodesic. However, in general cases the smooth solutions to (18) can not exist globally in time due to the appearance of nonlinearity arising from the ambient curved spacetimes. For the theory on geodesic equations in curved spacetimes, we refer to the monograph by Chandrasekhar [5].

Remark 2.4. *If the background spacetime is flat, then the ambient connections vanish and the system (18) will go back to the equations studied in [15].*

3 Light-like extremal surfaces in Schwarzschild spacetime

In this section, we study the light-like extremal surfaces in Schwarzschild spacetime, which is stationary, spherically symmetric and asymptotically flat. In the spherical coordinates (τ, r, α, β) , the Schwarzschild metric \tilde{g} reads

$$ds^2 = - \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\alpha^2 + \sin^2 \alpha d\beta^2), \quad (22)$$

where m is a positive constant standing for the universe mass.

For the above Schwarzschild metric, the equations of motion for light-like extremal surfaces (18) reduce to the following form

$$\left\{ \begin{array}{l} \tau_{tt} + \frac{2m}{r(r-2m)} \tau_t r_t = 0, \\ r_{tt} - (r-2m) \sin^2 \alpha \beta_t^2 - (r-2m) \alpha_t^2 - \frac{m}{r(r-2m)} r_t^2 + \frac{m(r-2m)}{r^3} \tau_t^2 = 0, \\ \alpha_{tt} + \frac{2}{r} r_t \alpha_t - \frac{1}{2} \sin 2\alpha \beta_t^2 = 0, \\ \beta_{tt} + \frac{2}{r} r_t \beta_t + \frac{2 \cos \alpha}{\sin \alpha} \alpha_t \beta_t = 0. \end{array} \right. \quad (23)$$

The first and last equation in (23) can be easily integrated, i.e.,

$$\tau_t = \frac{E(\vartheta)}{1 - \frac{2m}{r}}, \quad (24)$$

$$\beta_t = \frac{L(\vartheta)}{r^2 \sin^2 \alpha}, \quad (25)$$

where $E(\vartheta)$ and $L(\vartheta)$ are two smooth functions and will be determined by the initial data. Substituting (25) into the third equation in (23) yields

$$r^4 \sin^2 \alpha \alpha_t^2 = -L^2(\vartheta) \cos^2 \alpha + K(\vartheta) \sin^2 \alpha, \quad (26)$$

where $K(\vartheta)$ is a nonnegative function. Combining (24)-(26) and the second equation in (23) leads to the following equation for r :

$$r_{tt} - \frac{m}{r(r-2m)} r_t^2 + \frac{mE^2(\vartheta)}{r(r-2m)} - \frac{r-2m}{r^4} (K(\vartheta) + L^2(\vartheta)) = 0. \quad (27)$$

The equations (24)-(26) are analogous in some manner to the equations for the motion of null strings (see e.g. [7]). However, it is worth pointing out that we need $\Delta(\vartheta) = 0$, instead of $g_{00} = g_{01} = 0$ in the null string theory, see also the following discussions.

By the initial data (3), the functions $E(\vartheta)$, $L(\vartheta)$ and $K(\vartheta)$ are in fact given by

$$E(\vartheta) = \psi_0(\vartheta) \left(1 - \frac{2m}{\varphi_1(\vartheta)}\right), \quad L(\vartheta) = \psi_3(\vartheta) \varphi_1^2(\vartheta) \sin^2(\varphi_2(\vartheta)) \quad (28)$$

and

$$K(\vartheta) = \varphi_1^4(\vartheta) [\psi_2^2(\vartheta) + \psi_3^2(\vartheta) \sin^2(\varphi_2(\vartheta)) \cos^2(\varphi_2(\vartheta))]. \quad (29)$$

Here the lower and upper indices are used interchangeably without ambiguity .

In order to solve the light-like extremal surfaces in Schwarzschild spacetime, it suffices to consider the Cauchy problem (24)-(29) and (3) under the assumption (20). By computation, we have

$$\Lambda(\vartheta) = - \frac{-\left(1 - \frac{2m}{\varphi_1}\right) \varphi'_0 \psi_0 + \left(1 - \frac{2m}{\varphi_1}\right)^{-1} \varphi'_1 \psi_1 + \varphi_1^2 \varphi'_2 \psi_2 + \varphi_1^2 \sin^2 \varphi_2 \varphi'_3 \psi_3}{-\left(1 - \frac{2m}{\varphi_1}\right) (\varphi'_0)^2 + \left(1 - \frac{2m}{\varphi_1}\right)^{-1} (\varphi'_1)^2 + \varphi_1^2 (\varphi'_2)^2 + \varphi_1^2 \sin^2 \varphi_2 (\varphi'_3)^2}, \quad (30)$$

and

$$\begin{aligned} \Delta(0, \theta) &= \Delta(0, \vartheta) \\ &= (\psi_0 \varphi'_1 - \psi_1 \varphi'_0)^2 + \left(1 - \frac{2m}{\varphi_1}\right) \varphi_1^2 (\psi_0 \varphi'_2 - \varphi'_0 \psi_2)^2 \\ &\quad + \left(1 - \frac{2m}{\varphi_1}\right) \varphi_1^2 \sin^2 \varphi_2 (\psi_0 \varphi'_3 - \psi_3 \varphi'_0)^2 - \left(1 - \frac{2m}{\varphi_1}\right)^{-1} \varphi_1^2 (\psi_1 \varphi'_2 - \psi_2 \varphi'_1)^2 \\ &\quad - \left(1 - \frac{2m}{\varphi_1}\right)^{-1} \varphi_1^2 \sin^2 \varphi_2 (\psi_1 \varphi'_3 - \psi_3 \varphi'_1)^2 - \varphi_1^4 \sin^2 \varphi_2 (\psi_3 \varphi'_2 - \psi_2 \varphi'_3)^2, \end{aligned} \quad (31)$$

which will be useful in the following discussions.

Of course, this kind of Cauchy problem looks very complicated and can not be solved generally. So in what follows we would like to investigate some particular solutions to illustrate our method.

Example 1: We select the initial data as follows

$$t = 0 : \quad \varphi = (\tau_0, r_0, \alpha_0(\vartheta), \vartheta), \quad \psi = \left(\pm \left(1 - \frac{2m}{r_0}\right)^{-1} r_1, r_1, 0, 0 \right), \quad (32)$$

where τ_0, r_0, r_1 are constants and $\alpha_0(\vartheta)$ is an arbitrary function of ϑ . Further we suppose that $r_0 > 2m$. For this kind of initial data, direct computation shows that the assumptions (20) are satisfied and $\Delta(0, \vartheta) = 0$. By (28)-(29) and (32), it is easy to see that

$$L(\vartheta) = 0, \quad K(\vartheta) = 0, \quad E(\vartheta) = \pm r_1. \quad (33)$$

Then it follows from (25)-(26) and (32)-(33) that

$$\alpha = \alpha_0(\vartheta), \quad \beta = \vartheta. \quad (34)$$

Moreover, we can obtain from (27) the following equation for r :

$$r_{tt} - \frac{m}{r(r-2m)} r_t^2 + \frac{m r_1^2}{r(r-2m)} = 0, \quad (35)$$

where we have made use of (32) and (33). It can be observed that the solution to (35) is given by

$$r = \pm r_1 t + r_0. \quad (36)$$

Thus, by integration we derive from (24) and (36) the following solution

$$r - r_0 + 2m \ln \frac{r - 2m}{r_0 - 2m} = \pm (\tau - \tau_0), \quad \alpha = \alpha_0(\vartheta), \quad \beta = \vartheta. \quad (37)$$

Here we would like to emphasize that the solution (37) is a generalization of the so-called *cone strings* in the null string theory, since the cone strings require the coordinate α is a constant (see e.g., [7]). ■

The previous discussions imply that for the initial data (32) satisfying the condition (20), the light-like extremal surface is uniquely given by (37). Of course, by the method in the present paper, one can choose other various initial data to construct the corresponding light-like extremal surfaces.

Now we do some manipulations for our problem before we construct other types of light-like extremal surfaces. For simplicity we assume that the initial datum

$$\psi_3(\vartheta) \equiv 0, \quad (38)$$

which implies

$$L(\vartheta) \equiv 0. \quad (39)$$

By (25), it leads to

$$\beta = \varphi_3(\vartheta) \quad (40)$$

and the equation (26) becomes

$$\alpha_t = \pm \frac{\sqrt{K}}{r^2}, \quad (41)$$

where we have supposed that $\sin \alpha \neq 0$.

Let

$$z = r_t. \quad (42)$$

When $r_t \neq 0$, it follows from the third equation in (27) that

$$\frac{dz}{dr} = \frac{m}{r(r-2m)}z + \left[\frac{r-2m}{r^4}K - \frac{mE^2}{r(r-2m)} \right] \frac{1}{z}, \quad (43)$$

which in turn implies that

$$\frac{d}{dr} \left(\frac{z^2}{\frac{r-2m}{r}} \right) = \frac{d}{dr} \left(-\frac{K}{r^2} + \frac{2mE^2}{r-2m} \right). \quad (44)$$

Integrating (44) and noting the initial data, we have

$$r_t^2 = \left(C - \frac{K}{r^2} \right) \left(1 - \frac{2m}{r} \right) + \frac{2m}{r} E^2, \quad (45)$$

where

$$C = \frac{\psi_1^2 \varphi_1^3 + K(\varphi_1 - 2m) - 2mE^2 \varphi_1^2}{\varphi_1^2(\varphi_1 - 2m)}. \quad (46)$$

By considering r as a function of α (instead of t), we obtain from the equation (41) and (45) the following equation

$$\left(\frac{du}{d\alpha} \right)^2 = 2mu^3 - u^2 + 2mAu + B \triangleq g(u), \quad (47)$$

where $u = \frac{1}{r}$ and

$$A = -\frac{1}{\varphi_1^2} + \frac{(\varphi_1 - 2m)^2 \psi_0^2 - \varphi_1^2 \psi_1^2}{(\varphi_1 - 2m) \varphi_1^5 \psi_2^2}, \quad (48)$$

$$B = \frac{1}{\varphi_1^2} + \frac{-2m(\varphi_1 - 2m)^2 \psi_0^2 + \varphi_1^3 \psi_1^2}{(\varphi_1 - 2m) \varphi_1^6 \psi_2^2}. \quad (49)$$

Once equation (47) has been solved for $u(\alpha)$, the solution can be completed by direct quadratures of the following equations

$$\frac{dt}{d\alpha} = \pm \frac{1}{\sqrt{K}u^2} \quad \text{and} \quad \frac{d\tau}{d\alpha} = \mp \frac{E}{\sqrt{K}u^2(2mu - 1)}. \quad (50)$$

Example 2: We choose the following initial data,

$$t = 0 : \quad \varphi = (\tau_0, r_0, \alpha_0, \vartheta), \quad \psi = \left(f(\vartheta), 0, \pm \frac{1}{r_0^2} \sqrt{r_0(r_0 - 2m)f^2(\vartheta)}, 0 \right), \quad (51)$$

where τ_0, r_0, α_0 are constants and $r_0 > 2m$, $f(\vartheta)$ is a non-zero smooth function. For the above initial data, it is easy to see that the condition (20) and $\Delta(0, \vartheta) = 0$ can be satisfied. Meanwhile, by (28)-(29) and (51), we can observe that

$$L(\vartheta) = 0, \quad K(\vartheta) = r_0(r_0 - 2m)f^2(\vartheta) > 0, \quad E(\vartheta) = \left(1 - \frac{2m}{r_0} \right) f(\vartheta). \quad (52)$$

Moreover, by the initial data (51) we observe that the equation (47) can be rewritten as

$$\left(\frac{du}{d\alpha}\right)^2 = 2mu^3 - u^2 + \frac{r_0 - 2m}{r_0^3} = g(u). \quad (53)$$

Obviously, the solution relates to the disposition of the roots of the cubic equation $g(u) = 0$.

In fact, we have

$$g(u) = 2m(u - u_1)(u - u_2)(u - u_3), \quad (54)$$

where

$$u_1 = \frac{1}{r_0}, u_2 = \frac{r_0 - 2m + \sqrt{(r_0 - 2m)(r_0 + 6m)}}{4mr_0}, u_3 = \frac{r_0 - 2m - \sqrt{(r_0 - 2m)(r_0 + 6m)}}{4mr_0}. \quad (55)$$

It is noted from (55) that the third root u_3 of $g(u)$ is always negative.

Returning to equation (53), we will study the solution in the following several cases.

Case I: $r_0 = 3m$, equivalently, $u_1 = u_2 = \frac{1}{3m}$. The equation (53) becomes

$$\left(\frac{du}{d\alpha}\right)^2 = 2m\left(u + \frac{1}{6m}\right)\left(u - \frac{1}{3m}\right)^2. \quad (56)$$

Obviously, $u = \frac{1}{r_0} = \frac{1}{3m}$ is a solution to (56) satisfying the initial data. Then by (50) we can easily obtain the following light-like extremal surface

$$\tau = f(\vartheta)t + \tau_0, \quad r = 3m, \quad \alpha = \pm \frac{\sqrt{f^2(\vartheta)}}{3\sqrt{3m}}t + \alpha_0, \quad \beta = \vartheta. \quad (57)$$

This special solution has been discovered in [7] and we will not discuss it further on.

Case II: $2m < r_0 < 3m$, equivalently, $u_1 > u_2$. Bearing in mind that u initially equals to u_1 and the domain of u should be chosen so that $g(u) \geq 0$, we should consider the solution in the range $u_1 \leq u < \frac{1}{2m}$ in this case. We now make the substitution

$$u = u_2 + (u_1 - u_2) \sec^2 \frac{\xi}{2}. \quad (58)$$

It is noted from (58) that $\xi = 0$ when $u = u_1$. Substituting (58) into (53) gives

$$\left(\frac{d\xi}{d\alpha}\right)^2 = 2m(u_1 - u_3) \left(1 - k^2 \sin^2 \frac{\xi}{2}\right), \quad k^2 = \frac{u_2 - u_3}{u_1 - u_3} \quad (0 < k^2 < 1). \quad (59)$$

Then α can be solved in terms of the Jacobian elliptic integral,

$$\alpha = 2[2m(u_1 - u_3)]^{-\frac{1}{2}} F\left(\frac{\xi}{2}, k\right) + \alpha_0, \quad (60)$$

where

$$F(\chi, k) = \int_0^\chi \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}}. \quad (61)$$

In the derivation of (60), we have made use of the initial data (51).

Case III: $r_0 > 3m$, equivalently, $u_1 < u_2$. Noting the initial data, we have to study the solution in the range $0 < u \leq u_1$ for the present case. We can choose the following substitution

$$u = u_3 + \frac{1}{2}(u_1 - u_3)(1 - \cos \xi). \quad (62)$$

For this selection, we have $u = u_1$ when $\xi = \pi$. By this substitution, the equation (53) reduces to

$$\left(\frac{d\xi}{d\alpha}\right)^2 = 2m(u_2 - u_3) \left(1 - k^2 \sin^2 \frac{\xi}{2}\right), \quad k^2 = \frac{u_1 - u_3}{u_2 - u_3} \quad (0 < k^2 < 1). \quad (63)$$

Similarly, α can be expressed by the Jacobian elliptic integral as

$$\alpha = 2[2m(u_2 - u_3)]^{-\frac{1}{2}} F\left(\frac{\xi}{2}, k\right) + C_0, \quad (64)$$

where C_0 is determined by the initial data in the following form

$$C_0 = \alpha_0 - 2[2m(u_2 - u_3)]^{-\frac{1}{2}} F\left(\frac{\pi}{2}, k\right). \quad (65)$$

Thus, we have completed the discussions for this example. ■

Example 3: We now turn to consider the following initial data

$$t = 0 : \quad \varphi = (\varphi_0(\vartheta), \varphi_1(\vartheta), \varphi_2(\vartheta), \beta_0), \quad \psi = (\varphi'_0(\vartheta), \varphi'_1(\vartheta), \varphi'_2(\vartheta), 0), \quad (66)$$

which implies that

$$\Lambda(\vartheta) = -1, \quad \Delta(\vartheta) = 0 \quad (67)$$

and the condition (20) are satisfied automatically.

By the definition of $\Lambda(\vartheta)$, we need to require $g_{11} \neq 0$, which yields that the coefficient of u in (47) should be a non-zero constant. For brevity we assume that the constant

$$B = \frac{1}{\varphi_1^2} + \frac{-2m(\varphi_1 - 2m)^2 \psi_0^2 + \varphi_1^3 \psi_1^2}{(\varphi_1 - 2m)\varphi_1^6 \psi_2^2} = 0, \quad (68)$$

equivalently,

$$(\varphi'_2)^2 = \frac{2m(\varphi_1 - 2m)^2 (\varphi'_0)^2 - \varphi_1^3 (\varphi'_1)^2}{(\varphi_1 - 2m)\varphi_1^4}, \quad (69)$$

where we have made use of the initial data (66). Then A becomes a non-zero constant and in turn the induced metric components $g_{00} = g_{01} = g_{11} \neq 0$.

Remark 3.1. *The above argument shows that for some light-like extremal surfaces, each induced metric component may be non-zero. This important feature is very different from that of null strings, since for null strings, one requires $g_{00} = g_{01} = 0$ (see e.g., [7]).*

Next we discuss the light-like extremal surface in detail. Here for illustration purpose, we only consider the case that

$$\varphi_0 = \vartheta, \quad \varphi_1 = r_0 \quad (70)$$

and then from (69) we obtain φ_2 as

$$\varphi_2 = \pm \frac{\sqrt{2m(r_0 - 2m)}}{r_0^2} \vartheta. \quad (71)$$

So the initial data now become

$$t = 0 : \quad \varphi = (\vartheta, r_0, \pm \frac{\sqrt{2m(r_0 - 2m)}}{r_0^2} \vartheta, \beta_0), \quad \psi = (1, 0, \pm \frac{\sqrt{2m(r_0 - 2m)}}{r_0^2}, 0). \quad (72)$$

Meanwhile, we have

$$L = 0, \quad E = 1 - \frac{2m}{r_0}, \quad K = 2m(r_0 - 2m) > 0 \quad (73)$$

and the equation (47) can be rewritten as

$$\left(\frac{du}{d\alpha} \right)^2 = 2mu^3 - u^2 + \frac{r_0 - 2m}{r_0^2} u = 2mu \left(u - \frac{1}{r_0} \right) \left(u - \frac{r_0 - 2m}{2mr_0} \right). \quad (74)$$

In a similar manner to Example 2, we divide the arguments into several cases according to the values of r_0 .

Case I: $r_0 = 4m$, equivalently, $\frac{1}{r_0} = \frac{r_0 - 2m}{2mr_0}$. For this case, we can easily obtain the following simple solution

$$\tau = t + \vartheta, \quad r = 4m, \quad \alpha = \pm \frac{1}{8m}(t + \vartheta), \quad \beta = \beta_0. \quad (75)$$

Case II: $r_0 > 4m$, equivalently, $\frac{1}{r_0} < \frac{r_0 - 2m}{2mr_0}$. We only need to investigate the solution in the range $0 < u \leq \frac{1}{r_0}$. Take the following substitution

$$u = \frac{1}{2r_0}(1 - \cos \xi). \quad (76)$$

We note that $u = \frac{1}{r_0}$ when $\xi = \pi$. So the equation (74) reduces into

$$\left(\frac{d\xi}{d\alpha} \right)^2 = \frac{r_0 - 2m}{r_0} \left(1 - k^2 \sin^2 \frac{\xi}{2} \right), \quad k^2 = \frac{2m}{r_0 - 2m} \quad (0 < k^2 < 1) \quad (77)$$

and then α can be solved by

$$\alpha = 2 \left(\frac{r_0}{r_0 - 2m} \right)^{\frac{1}{2}} F \left(\frac{\xi}{2}, k \right) + C_0, \quad (78)$$

where C_0 is the integration constant,

$$C_0 = -2 \left(\frac{r_0}{r_0 - 2m} \right)^{\frac{1}{2}} F \left(\frac{\pi}{2}, k \right) \pm \frac{\sqrt{2m(r_0 - 2m)}}{r_0^2} \vartheta.$$

Case III: $2m < r_0 < 4m$, equivalently, $\frac{1}{r_0} > \frac{r_0 - 2m}{2mr_0}$. For this case, we will consider the solution in the range $\frac{1}{r_0} \leq u < \frac{1}{2m}$ and make the following substitution

$$u = \frac{r_0 - 2m}{2mr_0} + \frac{4m - r_0}{2mr_0} \sec^2 \frac{\xi}{2}, \quad (79)$$

where $u = \frac{1}{r_0}$ when $\xi = 0$. By this substitution, the equation (74) becomes

$$\left(\frac{d\xi}{d\alpha} \right)^2 = \frac{2m}{r_0} \left(1 - k^2 \sin^2 \frac{\xi}{2} \right), \quad k^2 = \frac{r_0 - 2m}{2m} \quad (0 < k^2 < 1). \quad (80)$$

Then α can be expressed by the Jacobian elliptic integral as

$$\alpha = 2 \left(\frac{r_0}{2m} \right)^{\frac{1}{2}} F \left(\frac{\xi}{2}, k \right) \pm \frac{\sqrt{2m(r_0 - 2m)}}{r_0^2} \vartheta. \quad (81)$$

Thus, we have finished the discussions for this example. ■

4 Concluding remarks

This paper proposes an effective method to study the light-like extremal surfaces in curved spacetimes, namely, one can carry out the analysis for light-like extremal surfaces by considering a Cauchy problem associated with a constraint (20) on the initial data. By developing a diffeomorphic mapping, this kind of Cauchy problem for light-like extremal surfaces can be transformed into an initial value problem for a kind of geodesic equations. Interestingly, by this method we succeed in obtaining many light-like extremal surfaces in Schwarzschild spacetime. Some of them are similar to the known results in null string theory and others are very new to our best knowledge.

As in [21], we believe that the equations (2) can be used to investigate the time-like extremal surfaces and space-like surfaces as well as light-like surfaces. The results in this paper verify this statement in some sense. At the same time, the Burgers equation (10) plays an important role in our analysis and can be viewed as the limiting case by

comparing with the key equations (3.18) in [21]. Finally, it will be interesting to apply the method in this paper to consider the light-like surfaces in Kerr spacetime or other important spacetimes.

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